A Quasi-Linear Free Surface Condition in Slow Ship Theory

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1. INTRODUCTION

It is well known in slow ship theory that the usual linearized free surface condition is not applicable when obstacles are considered which beam length ratio (B/L) is not sufficiently small. In that case, the linearized theory leads to the so-called "low Froude number paradox" for small values of the Froude number ($Fn^2 = U^2/gL << 1$), as shown f.i. by Salvesen [12] and Dagan [3]. The assumption that the velocity field is only a small perturbation of the unperturbed incoming field does not hold good then.

This problem has been studied by several authors resulting in proposals for a quasi-linear free surface condition, based on the assumption that not the unperturbed incoming velocity field, but the so-called "double body solution" (rigid wall solution) has to be considered as a first approximation for the velocity field. That assumption, confirmed by observations from experiments (f.i. Baba [1]), can also be made plausible with mathematical arguments, based on asymptotic analysis (with the Froude number as the small parameter), as shown by Hermans [6] and [7]. That approach is also followed in this paper, but the free surface condition derived here will also contain terms concerning the first order derivatives of the perturbation potential along the free surface. These terms, which are neglected by authors like Ogilvie [11], Maruo [9] and [10], and Baba [2] are similar to the ones in the work of Eggers [4]. It will be shown, for the two-dimensional case, that these terms have considerable influence on the behaviour of the solutions and may therefore not be removed from the free surface condition.

In section 2 the nonlinear problem is formulated and a "boundary-layer" approach is shown, resulting in a superposition of the double body solution and a perturbation potential as a first approximation for the wave solution of the total problem.

In section 3 a quasi-linear boundary condition is derived for this perturbation problem. The two-dimensional problem is treated as an example in section 4. In

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order to show the influence of the terms mentioned above, a more or less generalized free surface condition is used there.

In section 5 some results for the two-dimensional case are shown. Also there, the effect of violating Laplace's equation, which is inherent to the derivation given in section 3, is shown to be negligible in the final results.

Finally some conclusions are presented in section 6.

2. THE NONLINEAR PROBLEM AND THE BOUNDARY-LAYER APPROACH

An obstacle (ship) moves with velocity U at the free surface of an infinitely deep fluid. A cartesian coordinate system is chosen fixed on the body. With respect to this system the body is at rest and seems to be placed in a uniform stream with U as the magnitude of the velocity at infinity. A steady wave pattern can be observed attached to the body. The x-axis is chosen along the free surface at rest in the direction of the unperturbed incoming velocity field. With the y-axis perpendicular to the free surface at rest, and positive in upward direction, the z-axis has to be chosen along the free surface (see Fig. 1).

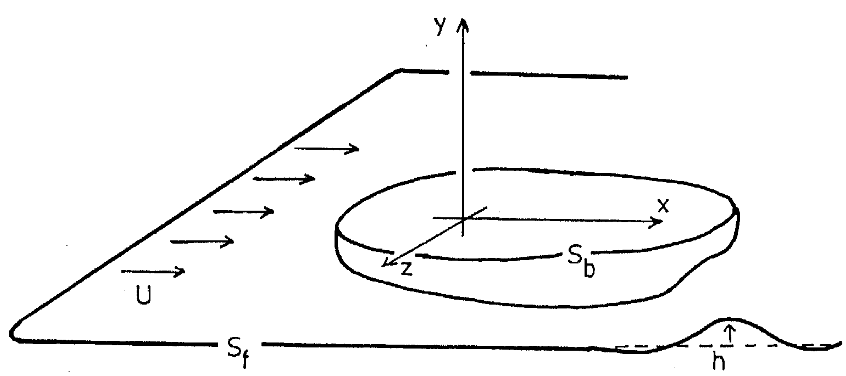


fig.1

The position of the free surface S_f is given by:

$$y = h(x,z) \tag{2.1}$$

The fluid is assumed to be inviscid and incompressible. A velocity potential $\Phi(x,y,z)$ may then be introduced, related to the velocity vector $\underline{u}$ by:

$$\underline{u} = \text{grad}\Phi \tag{2.2}$$

The law of continuity gives:

$$\Delta\Phi = 0 \quad \text{for} \quad y \le h(x,z) \quad \text{(outside the body)} \tag{2.3}$$

On the body's boundary S_b the kinematic condition holds:

$$\underline{\nabla}\Phi \cdot \underline{n} = 0 \quad \text{on } S_b \tag{2.4}$$

On the free surface both a kinematic- and a dynamic condition have to be fulfilled. Disregarding surface-tension effects, these two nonlinear conditions are given by:

$$\Phi_x h_x + \Phi_z h_z - \Phi_y = 0 \quad \text{on} \quad y = h(x,z) \tag{2.5}$$

$$\frac{1}{2} U^2 = gh + \frac{1}{2}\left(\Phi_x^2 + \Phi_y^2 + \Phi_z^2\right) \qquad \text{on} \quad y = h(x,z) \tag{2.6}$$

The condition at infinity, which is:

$$\Phi \to Ux + \text{"wave solution"} \quad \text{for } |\underline{x}| \to \infty \tag{2.7}$$

has to be combined with a proper radiation condition, which states that no waves are permitted far in front of the body.

For a fixed scale L (which depends on the body's geometry), small values of the Froude number coincide with small velocities U.
A first asymptotic expansion for Φ and h can be made:

$$\begin{cases} \Phi = \Phi_0 + \Phi_1 + \Phi_2 + \dots \\ h = h_0 + h_1 + h_2 + \dots \end{cases} \tag{2.8}$$

with $\Phi_i \sim o(\Phi_{i-1})$ and $h_i \sim o(h_{i-1})$ for $U \to 0$.

With $\Phi \sim O(U)$ as a consequence of the conditions at infinity it follows from (2.6) that $h \sim O(U^2)$. Using Taylor-expansions around $y = 0$ the following problems can be formulated, after substitution of the expansions of (2.8):

$$\begin{cases} \Delta\Phi_0 = 0 & y \le 0 \\ \Phi_{0y} = 0 & y = 0 \\ \underline{\nabla}\Phi_0 \cdot \underline{n} = 0 & \text{on } S_b \\ \Phi_0 \to Ux & \text{for } |\underline{x}| \to \infty \end{cases} \tag{2.9}$$

$$i \ge 1 \begin{cases} \Delta\Phi_i = 0 & y \le 0 \\ \Phi_{iy} = P_i(x,z) & y = 0 \\ \underline{\nabla}\Phi_i \cdot \underline{n} = 0 & \text{on } S_b \\ \Phi_i \to 0 & \text{for } |\underline{x}| \to \infty \end{cases} \tag{2.10}$$

The functions $h_i(x,z)$ may be computed afterwards, while the functions $P_i(x,z)$ contain terms which may be computed from the lower order terms of the expansions. Because of the fact that the second order derivatives have disappeared from the free surface condition, the radiation condition had to be dropped. The solutions constructed in this way are "non-wave" solutions. The problem turned out to be a singular perturbation problem. There is need for a boundary-layer approach, defining a thin layer beneath the free surface of order $O(Fn^2)$, in which a wave solution has to be constructed.

The expansions of (2.8) may be used to construct an "outer"-solution.
In [6] and [7] Hermans worked out a matching procedure with the following results: Firstly, in order to get a well-posed boundary value problem for the "inner"-solution a mixed approach has to be used:

$$\Phi^{inner} = \Phi^{outer} + \phi \tag{2.11}$$

with ϕ a perturbation potential only valid in the boundary-layer.
Secondly, only Φ_0 can be incorporated in the "outer"-solution (in contrary to the ideas of Keller [8]), because this mixed approach leads to boundary value problems for the functions Φ_i $(i \ge 1)$, different from the ones in (2.10).

The solution for Φ_0 (problem (2.9)) is the well known double body solution, which will be denoted by $\varphi_r(x,y,z)$ from now on.
The matching condition, using $\varphi_{ry}(x,0,z) = 0$, now takes the form:

$$\phi_y \to 0 \quad \text{outside the boundary layer} \tag{2.12}$$

which is a natural condition for a wave solution.

3. DERIVATION OF A QUASI-LINEAR FREE SURFACE CONDITION

The introduction of a boundary layer suggests that a coordinate-stretching should be carried out. However, this is not done here in order to avoid misinterpretations of the order of magnitude of the derivatives of φ_r.
The only assumption for the perturbation potential made beforehand is, that $\phi \sim o(\varphi_r)$ for $U \to 0$. Quadratic terms in ϕ may then be neglected after the substitution:

$$\Phi = \varphi_r + \phi \tag{3.1}$$

in the expressions of section 2.
For the dynamic free surface condition (2.6) this results in:

$$h(x,z) = \frac{1}{2g}\{U^2 - \varphi_{rx}^2 - \varphi_{ry}^2 - 2\varphi_{rx}\phi_x - 2\varphi_{ry}\phi_y - 2\varphi_{rz}\phi_z\} \quad \text{on } y = h(x,z) \tag{3.2}$$

With $h \sim O(U^2)$ for $U \to 0$, at every place where φ_r occurs, a Taylor-expansion is used around $y = 0$ (φ_r is only valid for $y \leq 0$!). Taking only the lowest order terms into account, and using $\varphi_{ry}(x,0,z) = 0$:

$$h(x,z) = h_r(x,z) + h_w(x,z)$$

with

$$h_r(x,z) = \frac{1}{2g}\{U^2 - \varphi_{rx}^2(x,0,z) - \varphi_{rz}^2(x,0,z)\} \tag{3.3}$$

$$h_w(x,z) = \frac{-1}{g}\{\varphi_{rx}(x,0,z)\phi_x(x,h,z) + \varphi_{rz}(x,0,z)\phi_z(x,h,z)\} \tag{3.4}$$

One should expect that now also a Taylor expansion for ϕ around $y = 0$ has to be made, in order to express $h(x,z)$ explicitly (as is done by Eggers [5]). However, there are two strong arguments against that approach. The first question is, whether it is asymptotically correct to truncate such an expansion after one or two terms. When a wave solution for ϕ is looked for, with wavenumber of $O(g/U^2)$, it is not clear beforehand, that for instance a term like $h^2\phi_{xyy}$ may be neglected compared with $h\phi_{xy}$. The second objection is related to the results achieved by this approach. For the two-dimensional case, the free surface condition given by Eggers [5] reads:

$$\phi_y + \frac{1}{g}\{[\frac{3}{2}\varphi_{rx}^2 - \frac{1}{2}U^2]\phi_{xx} + 3\varphi_{rx}\varphi_{rxx}\phi_x\} = D(x).$$

The point at the free surface at which $\varphi_{rx}^2 = \frac{1}{3} U^2$ is a singular point for this equation, because the coefficient of ϕ_{xx} becomes zero there. However, from physical point of view it looks rather strange that special attention should be paid to a point somewhere in the free surface flow before and behind the obstacle.

For that reason, a different approach is used here. Firstly, observing that h_w is proportional to ϕ, a Taylor-expansion is made around $y = h_r(x,z)$, after which a coordinate transformation is carried out:

$$\begin{cases} x' = x \\ y' = y - h_r(x,z) \\ z' = z \end{cases} \tag{3.5}$$

Taking only into account the lowest order terms, this gives for (3.4):

$$h_w(x', z') = -\frac{1}{g}\{ \varphi_{rx'} \phi_{x'} + \varphi_{rz'} \phi_{z'} \} \quad \text{on } y' = 0 \tag{3.6}$$

Finally, the whole procedure discussed above is repeated for the kinematic free surface condition (2.5). The expressions for h_r and h_w ((3.3) and (3.6)) are used, while for a convenient notation the accents for x, y and z have been dropped.

The final result is:

$$\begin{aligned} &\phi_y + \frac{1}{g}\{ \varphi_{rx}^2\phi_{xx} + 2\, \varphi_{rx}\varphi_{rz}\phi_{xz} + \varphi_{rz}^2\phi_{zz} \} + \\ &\frac{1}{g}\{ 3\varphi_{rx}\varphi_{rxx} + 2\varphi_{rz}\varphi_{rxz} + \varphi_{rx}\varphi_{rzz} \}\, \phi_x + \\ &\frac{1}{g}\{ 3\varphi_{rz}\varphi_{rzz} + 2\varphi_{rx}\varphi_{rxz} + \varphi_{rz}\varphi_{rxx} \}\, \phi_z = D(x,z) \quad \text{on } y = 0 \end{aligned} \tag{3.7}$$

with:

$$D(x,z) = \frac{\partial}{\partial x}[\varphi_{rx}(x,0,z)h_r(x,z)] + \frac{\partial}{\partial z}[\varphi_{rz}(x,0,z)h_r(x,z)]$$

and $h_r(x,z)$ as in (3.3). The wave elevation $h_w(x,z)$ can be computed afterwards, when ϕ is known, by (3.6).

The quasi-linear free surface condition has to be combined with Laplace's equation, which is, in the lowest order tems, not affected by the coordinate transformation:

$$\Delta\phi = 0 \qquad y \le 0 \tag{3.8}$$

It should be stated here that in (3.8) higher order terms, resulting from this transformation are neglected. In section 5 it will be shown that, for the two-dimensional case, corrections can be made in order to fulfill the exact Laplace equation. These corrections lead only to negligible contributions to the wave resistance there.

The free surface condition (3.7) differs from the ones given by most other authors by taking into account the last two terms in the left-hand-side. After the experience with the two-dimensional case it may be concluded that these terms are essential for construction of the correct solution for ϕ, even when ϕ is a wave solution! It is expected that omittance of these terms leads to problems with the determination of higher order terms of ϕ, similar to those that arise in asymptotic analysis of ordinary differential equations, when secular terms are involved.

4. THE TWO-DIMENSIONAL PROBLEM

In this section a totally submerged cylindrical body of infinite length is considered. With respect to the original x-y coordinate system, the body is at rest, and placed in a uniform stream (see Fig. 2).

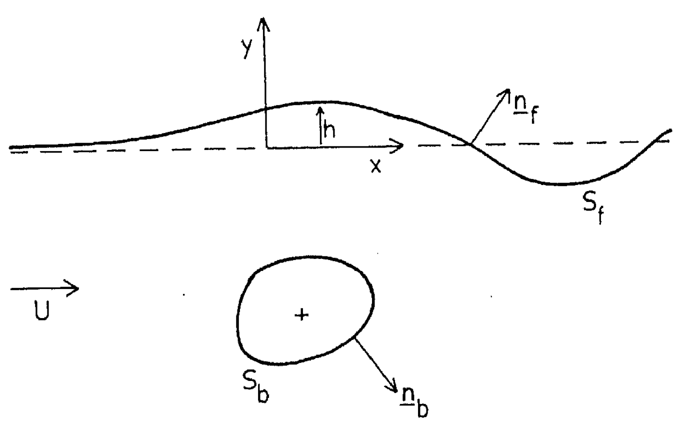


fig.2

The restriction is made for the body's centre to be situated as deep under the free surface, that the body's contour lies totally "outside" the boundary layer. The body's boundary condition, already satisfied by ϕ_r, may then be dropped from the problem for the perturbation potential ϕ.

With now x and y the transformed coordinates (see section 3), ϕ has to be a solution of:

$$\phi_{xx} + \phi_{yy} = 0 \qquad y \leq 0$$

$$\phi_y + \frac{1}{g}\{\varphi_{rx}^2\phi_{xx} + 3\varphi_{rx}\varphi_{rxx}\phi_x\} = D(x) \qquad \text{on} \quad y = 0$$

with

$$D(x) = \frac{\partial}{\partial x}[\, h_r(x)\varphi_{rx}(x,0)\,] \tag{4.1}$$

and

$$h_r(x) = \frac{1}{2g}\{\, U^2 - \varphi_{rx}^2(x,0)\,\}$$

$$\phi_x \to 0 \qquad \text{for} \qquad x \to -\infty$$

$\phi \to$ "wave-solution" for $x \to +\infty$

$\phi_y \to 0$ "outside" the boundary layer.

The wave profile is given by:

$$h_w(x) = -\frac{1}{g}\varphi_{rx}(x,0)\phi_x(x,0) \tag{4.2}$$

The free surface condition of (4.1) can be simplified after introduction of:

$$\lambda(x) = \frac{\varphi_{rx}^2(x,0)}{g} \tag{4.3}$$

In order to make clear the influence of the term $3\varphi_{rx}\varphi_{rxx}\phi_x$ in this condition, the factor 3 is replaced by 2α. The more general problem considered here is problem (4.1) in which the free surface condition is replaced by:

$$\phi_y + \lambda(x)\phi_{xx} + \alpha\lambda'(x)\phi_x = D(x) \quad \text{on } y = 0 \tag{4.4}$$

Keeping in mind that α should be 3/2.

The problem is solved with the help of Green's theorema, applied to the rectangular domain D_g of Fig. 3, with L_3 "outside" the boundary layer.

A Green's function $G(\xi, \eta; x,y)$ is introduced as a solution of Laplace's equation, representing a source of unity strength at $\xi = x$, $\eta = y$, where it behaves like:

$$G \sim \frac{1}{2\pi} \ln \left[(\xi - x)^2 + (\eta - y)^2\right]^{\frac{1}{2}} \tag{4.5}$$

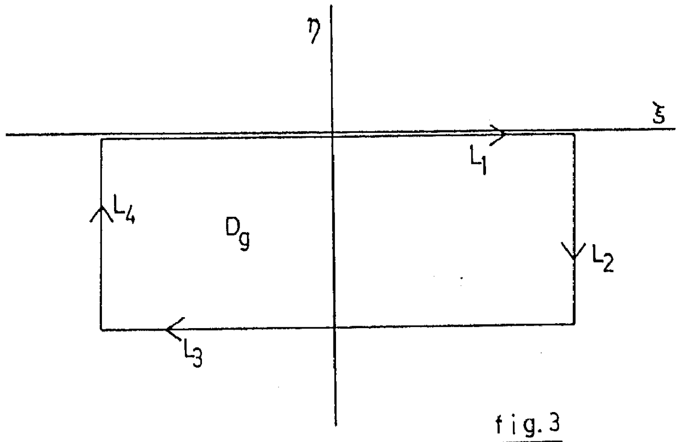


fig. 3

In order to get rid of integrals along L_2, L_3 and L_4, and part of the integral along L_1, a suitable problem to be considered for G is:

$$
\begin{aligned}
&G_{\xi\xi} + G_{\eta\eta} = 0 && \eta \leq 0 \\
&G_\eta + \frac{d^2}{d^2\xi}[\lambda G] - \alpha \frac{d}{d\xi}[\lambda' G] = 0 && \eta = 0 \\
&G \rightarrow \text{"wave solution"} && \xi \rightarrow -\infty \\
&G \rightarrow 0 && \xi \rightarrow +\infty \\
&G_\eta \rightarrow 0 \text{ "outside" the boundary layer} &&
\end{aligned} \tag{4.6}
$$

Notice, that G has to be a "wave-solution" for $\xi \rightarrow -\infty$, which will lead to the proper behaviour of ϕ for $x \rightarrow +\infty$.

Application of Green's theorema, with careful use of all the properties of ϕ and G, leads finally to:

$$c\phi(x,y) = -\int_{-\infty}^{+\infty} D(\xi) G(\xi,0;\ x,y)\, d\xi \tag{4,7}$$

with

$$\begin{cases} c = \frac{1}{2} & \text{for } y = 0 \\ c = 1 & \text{for } y < 0 \end{cases}$$

A solution for G may be constructed with the help of complex analysis. Taking:

$$G(\xi,\eta;x,y) = \mathrm{Re}[F(\zeta;z)] \tag{4.8}$$

with

$$\zeta = \xi + i\eta \quad \text{and } z = x + iy$$

and using a complex continuation for λ:

$$\lambda(\zeta) = \frac{1}{g}[\varphi_{r\xi}(\xi,\eta) - i\varphi_{r\eta}(\xi,\eta)]^2 \tag{4.9}$$

the function F has to be analytic in the lower half plane, except at $\zeta = z$, where:

$$F(\zeta;z) \sim \frac{1}{2\pi}\log(\zeta - z) \tag{4.10}$$

while at the real axis, F has to satisfy:

$$\mathrm{Im}\{i\frac{d^2}{d\zeta^2}[\lambda F] - \frac{d}{d\zeta}[F] - i\alpha\frac{d}{d\zeta}[\frac{d\lambda}{d\zeta}F]\} = 0 \tag{4.11}$$

F can be constructed by the same method as used in [7]. For this problem the result is given by:

$$F(\zeta;z) = \frac{1}{2\pi}\log\frac{(\zeta - z)}{(\zeta - \bar{z})} - \frac{\lambda^{\alpha-1}(\zeta)}{\pi i}\int_{+\infty}^{\zeta}\{\lambda^{-\alpha}(t)\log(t-\bar{z})\exp[i\int_{\zeta}^{t}\frac{ds}{\lambda(s)}]\}\,dt \tag{4.12}$$

in which the path of integration should be chosen as in Fig. 4 in order to get the proper wave behaviour for G.

Using (4.2) and (4.7) the expression for the wave elevation becomes:

$$h_w(x) = \frac{2\lambda^{\frac{1}{2}}(x)}{\sqrt{g}}\int_{-\infty}^{\infty}D(\xi)G_x(\xi,0;x,0)\,d\xi \tag{4.13}$$

From (4.12) it can be derived that:

$$G_x(\xi,0;x,0) = \mathrm{Re}\{\frac{-\lambda^{\alpha-1}(\xi)}{\pi i}\int_{\xi}^{\infty}\frac{\lambda^{-\alpha}(t)}{[t-(x+i.0)]}\exp[i\int_{\xi}^{t}\frac{ds}{\lambda(s)}]dt\} \tag{4.14}$$

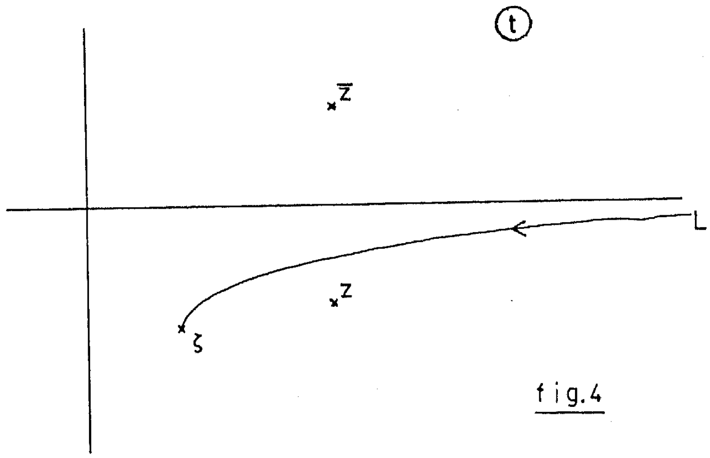
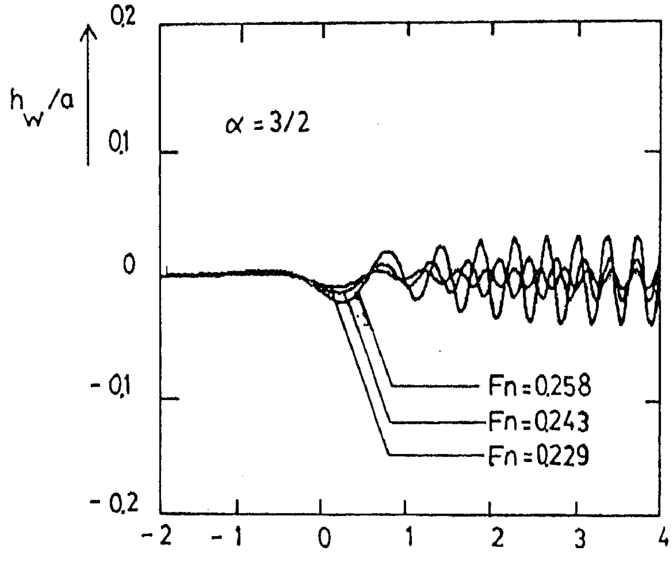


fig.4

The notation x + i.0 is used here, to show that the point approached the real axis from the upper half plane.
Although it can be shown that the wave amplitude, according to these expressions, is exponentially decaying with increasing wave number g/U^2, far behind the body, it is interesting to study the dependence on α of these solutions, as will be shown in the next section.

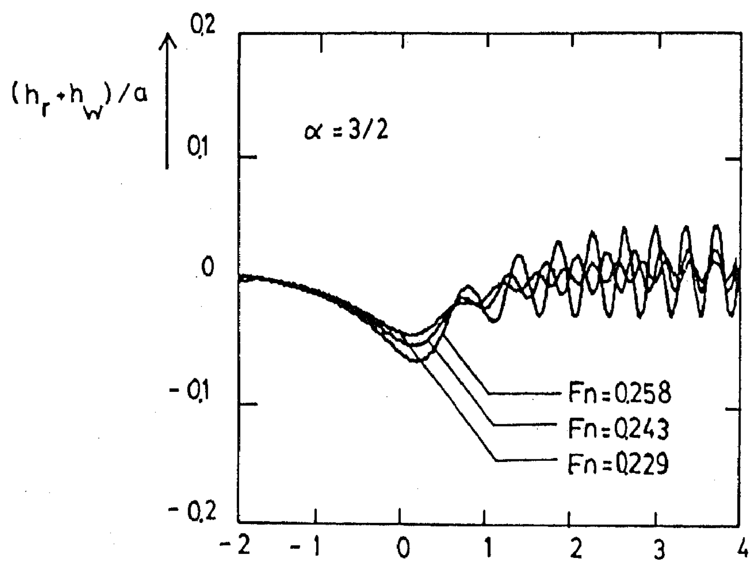
5. SOME RESULTS AND A CORRECTION TERM FOR LAPLACE'S EQUATION

It can be seen that the expression for G_x of (4.14) strongly depends on the choice made for the coefficient α. Especially when low values of λ occur at the free surface, the wave amplitude may differ in order of magnitude when different values of α are considered.
The results shown in this section, are those for a circular cylinder with its centre at $x = 0$, $y = -2a$.
The double body solution then may be obtained numerically; considering a source distribution at the cylinder's contour and at its mirror-image with respect to the x-axis. The functions $\lambda(x)$ and $D(x)$ can be calculated from this solution. The function G_x can also then be calculated numerically, with the neighbourhood of $t = x$ treated very carefully.
Results for the wave amplitude h_w are shown in Fig. 5, for $\alpha = 3/2$ and several values of the Froude number defined by $Fn = U^2/(ag)$.
The total free surface elevations $h_r + h_w$ are shown in Fig. 6.
In order to calculate the wave resistance, the function h_w has to be evaluated



0,2
h_w/a
0,1
$\alpha = 3/2$
0
- 0,1
- 0,2
Fn = 0,258
Fn = 0,243
Fn = 0,229
- 2
- 1
0
1
2
3
4
x/a

fig. 5



0,2
$(h_r + h_w)/a$
0,1
$\alpha = 3/2$
0
- 0,1
- 0,2
Fn = 0,258
Fn = 0,243
Fn = 0,229
- 2
- 1
0
1
2
3
4
x/a

fig. 6

for large values of x. For that aim the representation of G_x in (4.14) is not very useful. A more suitable representation can be found after construction of a Green's function, for the original problem (4.1) with $D(x) = 0$, $\tilde{G}(x,y;\xi,\eta)$, and using the symmetry principle of Green's functions: $G(\xi,\eta;x,y) = \tilde{G}(x,y;\xi,\eta)$. (See f.i. [7]).

The result for G_x:

$$G_x(\xi,0;x,0) = \mathrm{Re}\{ \frac{-\lambda^{-\alpha}(x)}{\pi i} \int_{-\infty}^{x} \frac{\lambda^{\alpha-1}(t)}{[t-(\xi+i.0)]} \exp[\, i \int_{x}^{t} \frac{ds}{\lambda(s)}\,]\, dt \} \qquad (5.1)$$

Now, for large values of x ($x \to \infty$) only the contribution of the pole at $t = \xi$ remains and:

$$G_x(\xi,0;x,0) \simeq -\,2\lambda^{-\alpha}(x)\,\lambda^{\alpha-1}(\xi)\,\mathrm{Re}\{\exp[i \int_{x}^{\xi} \frac{ds}{\lambda(s)}\,]\} \qquad (5.2)$$

Introducing a phase function:

$$S(x) = \int_{0}^{x} \frac{ds}{\lambda(s)} \qquad (5.3)$$

and observing that $S(x) \to \frac{g}{U^2}\,x$ for $x \to \infty$ because of $\lambda(x) \to \frac{U^2}{g}$, $h_w(x)$ is finally written as:

$$h_w(x) \simeq A \cos[\, k(x - c)\,] \qquad (5.4)$$

with $k = \frac{g}{U^2}$ and $A = \left| \frac{-4U}{g} \left(\frac{g}{U^2}\right)^{\alpha} \int_{-\infty}^{\infty} D(\xi)\,\lambda^{\alpha-1}(\xi)\exp[iS(\xi)]d\xi \right|$ for large values of x

The wave resistance coefficient, based on momentum analysis, is introduced as:

$$c_w = \frac{1}{4}\,\frac{ag}{U^2}\,A^2 \qquad (5.5)$$

with A as in (5.4).

The dependence of this coefficient on α is shown in Fig. 7, for the same circular cylinder, as a function of Fn.

Now some remarks should be made about the violation of Laplace's equation. As a consequence of the coordinate transformation the Laplace equation should read in the new coordinates (two-dimensional):

$$\phi_{xx} + \phi_{yy} + (h''_r\phi_y + 2h'_r\phi_{xy} + (h'_r)^2\phi_{yy}) = 0 \qquad y \le 0 \qquad (5.6)$$

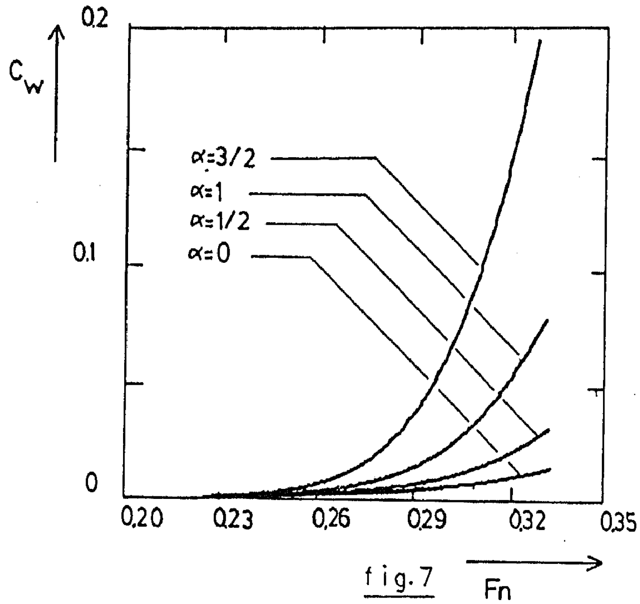


fig.7

In the problem solved for ϕ, (4.1), the term between brackets is neglected. Locally, this is justified by the fact that $h_r \sim O(U^2)$ and in the neglected term only second- and first order derivatives occur, which may be compared with the first part of the equation (which seems a better procedure than truncating a Taylor-expansion as shown in section 3).
Although it is locally a good argument to neglect this term, it has still to be shown that also its influence in the final results is small.
For that reason ϕ is written as:

$$\phi = \phi_0 + \phi_1 \tag{5.7}$$

With ϕ_0 the solution of problem (4.1), as given in section 4, resulting in a wave profile h_w as given in (4.13). In order to fulfill the exact Laplace equation (5.6), ϕ_1 has to be a solution of the equation:

$$\phi_{1xx} + \phi_{1yy} = \sigma(x,y) \qquad y \le 0 \tag{5.8}$$

with

$$\sigma(x,y) = - h_r''\phi_{0y} - 2\, h_r'\phi_{0xy} - (h_r')^2\phi_{0yy}.$$

The boundary conditions are the same as in 4.1, but now with $D(x) = 0$.

For this problem the same Green's function may be used with as result:

$$c\phi_1(x,y) = \iint_{Dg} \sigma(\xi,\eta)\, G(\xi,\eta;x,y)\, d\xi d\eta \tag{5.9}$$

with

$c = 1$ for $y < 0$

$c = \frac{1}{2}$ for $y = 0$ and G given by (4.8)

The correction for the wave height is:

$$h_{w1}(x) = \frac{-2\lambda^{\frac{1}{2}}(x)}{\sqrt{g}} \iint_{Dg} \sigma(\xi,\eta)\, G_x(\xi,\eta;x,0)\, d\xi d\eta \tag{5.10}$$

Observing that G_x (see (4.14)) is exponentially decaying in negative y-direction, the lowest order term of (5.10) may be obtained by partial integration with respect to η:

$$h_{w1}(x) \simeq \frac{2\lambda^{\frac{1}{2}}(x)}{\sqrt{g}} \int_{-\infty}^{\infty} \lambda(\xi)\, \sigma(\xi,0)\, G_x(\xi,0;x,0)\, d\xi \tag{5.11}$$

Because of $\lambda \sim O(\frac{1}{k})$, $\phi_0 \sim A(x)\exp[ikH(x)]$ (see previous section), hence $\sigma \sim kA\exp[ikH]$ and $G_x \sim O(k)$ (see (4.14)), the integrand is proportional to $kA\exp[ikH]$. Integration leads to:

$$h_{w1}(x) \sim \frac{1}{k} A \exp[ikH]. \tag{5.12}$$

With the same assumption for ϕ_0, the wave profile calculated in the previous section is estimated as:

$$h_{w0}(x) = \frac{-\lambda^{\frac{1}{2}}(x)}{\sqrt{g}} \phi_{0x} \sim A \exp[ikH] \tag{5.13}$$

According to this result it is concluded that the correction leads only to a higher order contribution for the wave amplitude, and may indeed be neglected.

6. CONCLUDING REMARKS

A quasi-linear free surface condition has been derived, valid for low Froude numbers. The double body potential was used as a first approximation for the flow field. The only restriction made for the perturbation potential was that quadratic terms were small enough to be neglected. The incorporation of terms with the first order derivatives of the perturbation potential, neglected by most other authors, did not make the analysis more difficult (at least for the two-dimensional case). In fact, they were essential for obtaining the correct order of magnitude of the wave solution, as can be seen in the results for the wave resistance in section 5. It is expected that these terms are also important in the three-dimensional case. For that reason, the free surface condition given in section 3 seems to be the best choice in slow ship theory. When the ray-method is applied for the three-dimensional case the above mentioned terms lead to some additional terms in the transport equations. However, the use of the ray-method in slow ship theory is yet not developed far enough to incorporate also the transport equations. At the moment it seems difficult enough to solve the eikonal equation in case of low Froude numbers.

Finally, it should be stated that the use of a coordinate transformation as carried out in section 3 is not only justified by the minor effects in the final results of the correction terms in section 5, but also by the fact that a singular point in the final free surface condition, is a stagnation point of the double body flow (when it occurs) at the free surface. From physical point of view this seems to be more acceptable than the occurence of such a singular point elsewhere which may result from the use of Taylor-expansions as mentioned in section 3.

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